STATS 207: Time Series Analysis
Autumn 2020

Lecture 14: State-Space Modelling and Kalman Filtering

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Genera Info

- HW3 is Due Monday 11/2/2020.

Outline

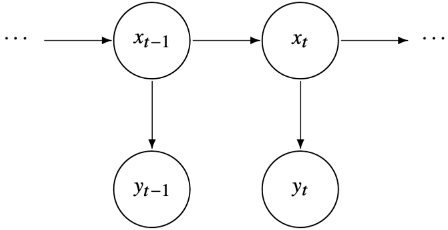
State-Space Models

Kalman Filtering

State-Space Models

Motivation

- Model time series as two different ones:
  1. **Unobserved state** (x_t).
  2. **Observations** (y_t).



- Separation of **state dynamics** and **observation procedure**.

State-Space Model – Definition

1. Definition: **State Equation**:

$$\boldsymbol{x}_t = \Phi \boldsymbol{x}_{t-1} + \boldsymbol{w}_t,$$

   where

   - $\boldsymbol{x}_t$, $\boldsymbol{w}_t$ are $p \times 1$,
   - Φ is $p \times p$,
   - $\boldsymbol{w}_t \overset{iid}{\sim} \mathcal{N}(0, Q)$.

2. Definition: **Observation Equation**:

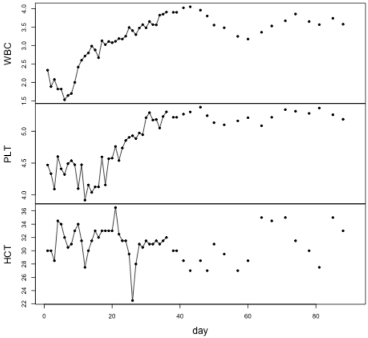
$$\boldsymbol{y}_t = A_t \boldsymbol{x}_t + \boldsymbol{v}_t$$

   - y_t, v_t are $q \times 1$,
   - A_t is $q \times p$,
   - $v_t \overset{iid}{\sim} \mathcal{N}(0, R)$.

- **Initial Conditions**: $\boldsymbol{x}_0 \sim \mathcal{N}(\mu_0, \Sigma_0)$.
- **Other names**: Linear Gaussian Model, Dynamic Linear Model (DLM).

Example 6.1: Biomedical Monitoring

```
plot(blood, type='o', pch=19, xlab='day', main='')
```



- 3 variables:
  1. log(white blood cell count) [WBC];
  2. log(platelet count) [PLT];
  3. Hematocrit level [HCT];
- New feature: **Missing values**.
- We can **model** all 3 variables using the SS approach and **estimate** the missing values.

Example 6.1: Biomedical Monitoring (cont'd)

- State Equation

$$\begin{bmatrix} x_{t1} \\ x_{t2} \\ x_{t3} \end{bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{12} & \phi_{13} \\ \phi_{21} & \phi_{22} & \phi_{23} \\ \phi_{31} & \phi_{32} & \phi_{33} \end{bmatrix} \begin{bmatrix} x_{t-1,1} \\ x_{t-1,2} \\ x_{t-1,3} \end{bmatrix} + \begin{bmatrix} w_{t1} \\ w_{t2} \\ w_{t3} \end{bmatrix}$$

- Observation Equation:

$$\boldsymbol{y}_t = A_t \boldsymbol{x}_t + v_t, \qquad A_t = \begin{cases} I_{3\times 3} & \text{observed,} \\ 0_{3\times 3} & \text{missing.} \end{cases}$$

- Covariances: Q and R are diagonal.

Example of State-Space Models

- **AR**(p) model:

$$x_t = \sum_{i=1}^{p} \phi_i x_{t-i} + w_t.$$

- **State Equation**:

$$\boldsymbol{x}_t = \Phi \boldsymbol{x}_{t-1} + \boldsymbol{w}_t,$$

where

  - $\boldsymbol{x}_t = (x_t, x_{t-1}, \ldots, x_{t-p+1})'$,
  - $\boldsymbol{w}_t = (w_t, 0, , \ldots, 0)'$ ($Q_{1,1} = \sigma_w^2$, $Q_{i,j} = 0$ for $(i,j) \neq (1,1)$)
  - $$\Phi = \begin{bmatrix} \phi_1 & \phi_2 & \cdots & \phi_{p-1} & \phi_p \\ 1 & 0 \cdots & 0 & 0 & \\ 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

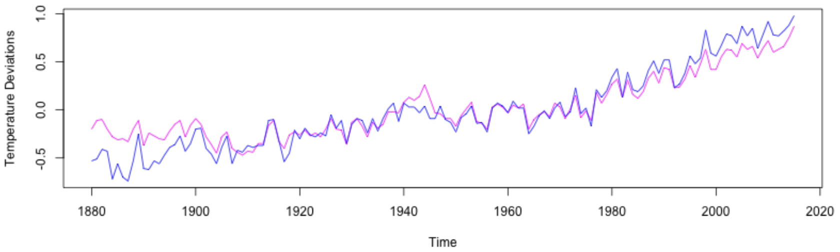
- **Observation Equation**:

$$y_t = \overbrace{(1, 0, \ldots, 0)'}^{A} \boldsymbol{x}_t + 0$$

($R = 0$ and $q = 1$)

Example 6.2: Global Warming, I

```
ts.plot(globtemp, globtempl, col=c(6,4), ylab='Temperature Deviations')
```



- Two global temperature series:
  1. Land-based.
  2. Marine Based.
- Suppose both series are observing the **same signal** with **different noises**

$$y_{t1} = x_t + v_{t1}, \qquad y_{t2} = x_t + v_{t2}.$$

- Common trend model: $x_t = x_{t-1} + w_t$.

Example 6.2: Global Warming, II

Represent in State-Space form:

- **State Equation**:

$$\boldsymbol{x}_t = \Phi \boldsymbol{x}_{t-1} + w_t$$

  - **State Vector**: $\boldsymbol{x}_t = [x_t]$ $(p = 1)$.
  - **State Dynamics**: $\Phi = [1]$.
  - $Q = \mathrm{Cov}([w_t]) = \sigma_w^2$.

- **Observation Equation**:

$$\boldsymbol{y}_t = \begin{bmatrix} y_{t1} \\ y_{t2} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} x_t + \begin{bmatrix} v_{t1} \\ v_{t2} \end{bmatrix}, \qquad R = \mathrm{Var}\left(\begin{bmatrix} v_{t1} \\ v_{t2} \end{bmatrix} \right) = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix}.$$

State-Space with Exogenous Variables

- Suppose that $(\boldsymbol{u}_t)$ is a $r \times 1$-dimensional vector series.
- **State-space model**:

$$\boldsymbol{x}_t = \Phi \boldsymbol{x}_{t-1} + \Upsilon \boldsymbol{u}_t + \boldsymbol{w}_t,$$

$$\boldsymbol{y}_t = A_t x_t + \Gamma \boldsymbol{u}_t + \boldsymbol{v}_t,$$

  where:
  - Υ is $p \times r$,
  - Γ is $q \times r$.
- Example 6.2 (Global Warming):
  - Suppose common trend+drift model:

$$x_t = \delta + x_{t-1} + w_t, \quad \delta > 0.$$

  - State-space equations:

$$\boldsymbol{x}_t = \boldsymbol{x}_{t-1} + \Upsilon \boldsymbol{u}_t + \boldsymbol{w}_t$$

    where $\boldsymbol{u}_t = [1]$ and $\Upsilon = \delta$.
  - The **observation equation** does not change ($\Gamma = 0$).

Example 6.3: AR(1) Process Hidden in Observation Noise, I

- **Univariate** SS model with **noisy** observations

$$y_t = x_t + v_t, \qquad v_t \stackrel{iid}{\sim} \mathcal{N}(0, \sigma_v^2),$$

and AR(1) state

$$x_t = \phi x_{t-1} + w_t, \qquad w_t \stackrel{iid}{\sim} \mathcal{N}(0, \sigma_w^2), \qquad x_0 \sim \mathcal{N}\left(0, \frac{\sigma_w^2}{1-\phi^2}\right).$$

- Warning: (y_t) is not AR(1)! (actually, it is ARMA$(1,1)$)

State-Space Model Representation of MA(1)

- Standard MA(1)

$$x_t = w_t + \theta w_{t-1}.$$

- **State**:

$$\boldsymbol{x}_t = [w_t, w_{t-1}]'.$$

- **State Equation**:

$$\boldsymbol{x}_t = \Phi \boldsymbol{x}_{t-1} + \boldsymbol{v}_t, \qquad \Phi = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \qquad Q = \begin{bmatrix} \sigma_w^2 & 0 \\ 0 & 0 \end{bmatrix}.$$

- **Observation Equation**:

$$\boldsymbol{y}_t = [1, \theta] \boldsymbol{x}_t.$$

Kalman Filtering

Why State-Space Models?

- Versatile Model
  - Time-varying/invariant observation A_t
  - Separation between observations and state dynamics.
- Missing data.
- Original motivation: **Kalman Filter**:
  - **Recursive** set of equations for obtaining **fundamental quantities at time t** from **fundamental quantities at time $t-1$**.
  - **Best prediction**:

$$\mathbf{x}_t^s \equiv \mathbb{E}\left[\mathbf{x}_t | \mathbf{y}_s, \mathbf{y}_{s-1}, \ldots\right].$$

  - **Error Covariance**:

$$P_{t_1,t_2}^s = \mathbb{E}\left[(\mathbf{x}_{t_1} - \mathbf{x}_{t_1}^s)(\mathbf{x}_{t_2} - \mathbf{x}_{t_2}^s)'\right].$$

  - Widely used in engineering applications.

Kalman Filter

A new approach to linear filtering and prediction problems

RE Kalman - 1960

The classical filtering and prediction problem is re-examined using the Bode-Shannon representation of random processes and the “state-transition” method of analysis of dynamic systems. New results are:(1) The formulation and methods of solution of the problem apply without modification to stationary and nonstationary statistics and to growing-memory and infinite-memory filters.(2) A nonlinear difference (or differential) equation is derived for the covariance matrix of the optimal estimation error. From the solution of this equation the co …

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https://www.youtube.com/watch?v=aNzGCMRnvXQ

Barack Obama Congratulating Rudolf Kalman

(October 7, 2009)

Bio Facts: Rudolf Kalman

Born	Rudolf Emil Kálmán[1] May 19, 1930 Budapest, Hungary
Died	July 2, 2016 (aged 86)[2] Gainesville, Florida
Citizenship	Hungary United States
Alma mater	Massachusetts Institute of Technology; Columbia University
Awards	IEEE Medal of Honor (1974) Rufus Oldenburger Medal (1976) Kyoto Prize (1985) Richard E. Bellman Control Heritage Award (1997) Charles Stark Draper Prize (2008) National Medal of Science (2009)
	Scientific career
Fields	Electrical Engineering; Mathematics; Applied Engineering Systems Theory
Institutions	Stanford University; University of Florida; Swiss Federal Institute of Technology
Doctoral advisor	John Ragazzini

Filtering, Smoothing, and Forecasting

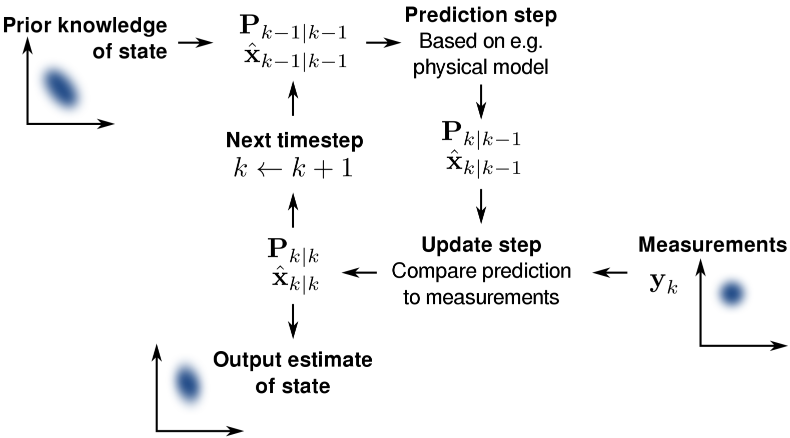
Terminology:

- $\boldsymbol{y}_{1:s} \equiv \{\boldsymbol{y}_1, \ldots, \boldsymbol{y}_s\}$.
- $\boldsymbol{x}_t^s \equiv \mathbb{E}[\boldsymbol{x}_t | \boldsymbol{y}_{1:s}]$ (**best** MSE estimator).
- Different problems:
    - **Prediction** or **Forecasting**: $s < t$.
    - **Filtering**: $s = t$.
    - **Smoothing**: $s > t$.

Kalman Filter (Property 6.1)

- **Initial Conditions**: $\boldsymbol{x}_0^0 = \mu_0,\ P_0^0 = \Sigma_0$.
- **Recursion** $t = 1, 2, \ldots, n$:
  - **State Prediction**:
  $$\boldsymbol{x}_t^{t-1} = \Phi \boldsymbol{x}_{t-1}^{t-1} + \Upsilon \boldsymbol{u}_t.$$
  - **State Covariance**:
  $$P_t^{t-1} \equiv \Phi P_{t-1}^{t-1} \Phi' + Q.$$
  - Observation **Prediction Errors**:
  $$\boldsymbol{\epsilon}_t \equiv \boldsymbol{y}_t - A_t \boldsymbol{x}_t^{t-1} - \Gamma \boldsymbol{u}_t.$$
  - **Kalman Gain**:
  $$K_t \equiv P_t^{t-1} A_t' \left(A_t P_t^{t-1} A_t' + R \right)^{-1}.$$
  - **State Estimation**:
  $$\boldsymbol{x}_t^t = \boldsymbol{x}_t^{t-1} + K_t \boldsymbol{\epsilon}_t.$$
  - **State Uncertainty**:
  $$P_t^t = (I - K_t A_t) P_t^{t-1}.$$

Signal Flow in Kalman Filter



Example 6.4: Random Walk in Noise

- Random walk in noise model:

$$x_t = x_{t-1} + w_t, \qquad x_0 = w_0 \sim \mathcal{N}(0,1),$$
$$y_t = x_t + v_t.$$

(1A) $x_t^{t-1} = x_{t-1}^{t-1}$.

(1B) $P_t^{t-1} = P_{t-1}^{t-1} + \sigma_w^2$.

(2A) $\epsilon_t = y_t - x_t^{t-1}$

(2B) $K_t = P_t^{t-1}/(P_t^{t-1} + \sigma_v^2)$.

(3A) $x_t^t = x_t^{t-1} + K_t \epsilon_t$.

(3B) $P_t^t = (1 - K_t) P_t^{t-1}$.

Kalman Smoother, I

- Kalman Filter is **Causal**: $\boldsymbol{x}_t^t$ only depends on observations until time t.
- Best **non-causal** Estimate:

$$\boldsymbol{x}_t^n = \mathbb{E}\left[\boldsymbol{x}_t | \boldsymbol{y}_{1:n}\right], \quad t < n.$$

- **Error** of Best **non-causal** Estimate:

$$P_t^n = \mathbb{E}\left[\left(\boldsymbol{x}_t - \boldsymbol{x}_t^n\right)\left(\boldsymbol{x}_t - \boldsymbol{x}_t^n\right)'\right].$$

Kalman Smoother, II

Property 6.2:

- **Initial Conditions**: $\boldsymbol{x}_n^n$, P_n^n (output of standard Kalman filter at time $t = n$).
- **Recursion** $t = n, n-1, \ldots, 1$:

$$\boldsymbol{x}_{t-1}^n = \boldsymbol{x}_{t-1}^{t-1} + J_{t-1}(\boldsymbol{x}_t^n - \boldsymbol{x}_t^{t-1})$$
$$P_{t-1}^n = P_{t-1}^{t-1} + J_{t-1}(P_t^n - P_t^{t-1})J_{t-1}',$$

where

$$J_{t-1} = P_{t-1}^{t-1}\Phi_t'[P_t^{t-1}]^{-1}.$$

Tracking Noisy Pointer Measurements

https://www.cs.utexas.edu/~teammco/misc/kalman_filter/