STATS 207: Time Series Analysis
Autumn 2020

Lecture 12: Spectral Analysis

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Genera Info

- HW3 is out. Due Monday 11/2/2020.
- Thank you for filling out **midquarter feedback**.

Spectral Analysis – So Far...

- **Periodogram** indicates the **component of data variance explainable by sinusoids** at frequency j.
- The **spectral density** $f(\omega)$ has a **Fourier series** representation with coefficients given by the **covariance function** $\gamma(h)$.
- The **spectral density** gives **typical size** of random variable **periodogram**.

- The **spectral density** and **cross-spectral density** play nicely with linear filtering.

Outline

Spectral Density

Linear Filters and Spectral Density

Cross-Spectra

Linear Filters and Cross Spectra

Spectral Estimation

Smoothing the Periodogram

Coherence Estimation

Frequency-domain Regression

Lagged Regression

Optimum Filtering

Spectral Density

Linear Filters and Spectral Density, I (review)

- Definition: **Linear filtering** of (x_t) to produce (y_t)

$$y_t = \sum_{j=-\infty}^{\infty} a_j x_{t-j}, \qquad \sum_{j=-\infty}^{\infty} |a_j| < \infty.$$

  "(y_t) is the **convolution** of x_t and (a_t)".

- Definition: $(a_t)_{t\in\mathbb{Z}}$ is the filter's **impulse response function**.
- Definition: The filter's **frequency response function** is

$$A(\omega) \equiv \sum_{j=-\infty}^{\infty} a_j e^{-2\pi i \omega j}.$$

- Property 4.3: If (x_t) has spectrum $f_x(\omega)$, then

$$f_y(\omega) = |A(\omega)|^2 f_x(\omega).$$

Linear Filters and Spectral Density, II (review)

- Example: Differencing

$$y_t = \nabla x_t$$

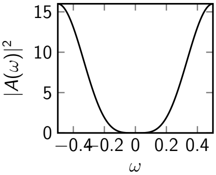
  - Frequency response

$$A(\omega) = 1 - e^{-2\pi i\omega}.$$

  - Relation between spectra

$$f_y(\omega) = |A(\omega)|^2 f_x(\omega) = \left|1 - e^{-2\pi i\omega}\right|^2 f_x(\omega) = 2\left(1 - \cos\left(2\pi\omega\right)\right)^2 f_x(\omega).$$

  - Example: x_t is white noise with intensity σ^2:
    $f_y(\omega) = |A(\omega)|^2\sigma^2 = 2\left(1 - \cos\left(2\pi\omega\right)\right)^2 \sigma^2$



"Differencing white noise creates a bluish noise."

Linear Filters and Spectral Density, III (review)

- Example: Symmetric Moving Average:

$$(a_t) = \left(\dots, 0, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, 0, \dots \right)$$

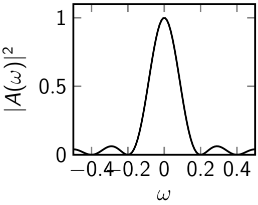
$$y_t = \sum_{j=-\infty}^{\infty} a_j x_{t-j} = \frac{1}{5} \left(x_{t-2} + x_{t-1} + x_t + x_{t+1} + x_{t+2} \right).$$

  - Frequency response:

$$A(\omega) = \frac{1}{5} \left[1 + 2\cos(2\pi\omega) + 2\cos(4\pi\omega) \right]$$

x_t is white noise of intensity σ^2:

$$f_y(\omega) = |A(\omega)|^2 \sigma^2$$



  - "Moving average of white noise creates a pinkish noise."

Cross-Spectra

Cross-Covariance

- **Recall**: The **cross-covariance** of two jointly stationary processes (x_t) and (y_t) is

$$\gamma_{xy}(h) = \mathrm{Cov}(x_{t+h}, y_t).$$

- Example: Delay + noise:

$$y_t = a \cdot x_{t-d} + w_t, \qquad (x_t), (w_t) \text{ are stationary and uncorrelated.}$$

$$\begin{aligned} \gamma_{xy}(h) &= \mathrm{Cov}(x_{t+h}, a \cdot x_{t-d} + w_t) \\ &= a \cdot \mathrm{Cov}(x_{t+h}, x_{t-d}) = a \cdot \gamma_x(h + d). \end{aligned}$$

Cross-Spectral Density

- Definition: For two **jointly stationary** processes (x_t) and (y_t), suppose that
$$\sum_{h=-\infty}^{\infty} |\gamma_{xy}(h)| < \infty.$$
Then the Fourier series
$$f_{xy}(\omega) = \sum_{h=-\infty}^{\infty} \gamma_{xy}(h) e^{-2\pi i \omega h},$$
defines a continuous **complex-valued** function on $(-1/2, 1/2)$, denoted the **cross-spectral density**.
- $\gamma_{xy}(h)$ can be recovered from
$$\gamma_{xy}(h) = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{2\pi i \omega h} f_{xy}(\omega) d\omega, \qquad h = 0, \pm 1, \pm 2, \ldots.$$
(Fourier coefficients of $f_{xy}(\omega)$)

Properties of Cross-Spectral Density

- Warning: $f_{xy}(\omega)$ is, in general, complex-valued.
- **Real/Imaginary Decomposition**:

$$f_{xy}(\omega) = \overbrace{c_{xy}(\omega)}^{\text{cospectrum}} - i \overbrace{q_{xy}(\omega)}^{\text{quadspectrum}}, \qquad \omega \in (-1/2, 1/2).$$

- Hermitian Symmetry:

$$f_{xy}(\omega) = f^*_{yx}(\omega),$$

$$c_{xy}(\omega) = c_{yx}(\omega), \qquad q_{xy}(\omega) = -q_{yx}(\omega).$$

(why?)

Coherence

- Definition: **Squared Coherence** function

$$\rho^2_{xy}(\omega) = \frac{|f_{yx}(\omega)|^2}{f_x(\omega)f_y(\omega)}$$

  (note similarity to correlation).
- Range:

$$0 \le \rho^2_{xy}(\omega) \le 1.$$

- Interpretation:
  - $\rho = 1$ implies **perfect correlation** at frequency ω.
  - $\rho = 0$ implies **uncorrelatedness** at frequency ω.
  - "If two processes are strongly coherent at ω, we can estimate the sinusiod component of frequency ω of (y_t) by observing (x_t).

Cross-Spectral Density, Example

Delay + noise:

$$y_t = x_{t-d} + w_t, \qquad (w_t) \text{ is uncorrelated with } (x_t).$$

- Cross-spectrum:

$$f_{yx}(\omega) = e^{-2\pi i d\omega} f_x(\omega).$$

- **Amplitude** of cross-spectrum:

$$|f_{yx}(\omega)| = |f_{xy}(\omega)| = |f_x(\omega)| = f_x(\omega).$$

Cross-Spectral Density, Example (cont'd)

Delay + noise:

$y_t = x_{t-d} + v_t, \qquad (v_t)$ stationary noise process uncorrelated with (x_t).

- Spectral density of (y_t):

$$f_y(\omega) = f_x(\omega) + f_v(\omega)$$

- Squared Coherence:

$$\rho^2_{xy}(\omega) = \frac{|f_{xy}(\omega)|^2}{f_x(\omega)f_y(\omega)} = \frac{|f_x(\omega)|^2}{f_x(\omega)f_y(\omega)} = \frac{f_x(\omega)}{f_x(\omega) + f_v(\omega)}$$

- **Signal-to-Noise Ratio** (SNR):

$$\text{SNR}(\omega) \equiv \frac{f_x(\omega)}{f_v(\omega)} \geq 0.$$

- Squared Coherence in terms of SNR:

$$\rho^2_{xy}(\omega) = \frac{\text{SNR}(\omega)}{1 + \text{SNR}(\omega)} \in [0, 1], \qquad \omega \in (-1/2, 1/2).$$

Linear Filters and Cross Spectra

- **Recall**: **Delay** + noise:

$$y_t = x_{t-d} + \text{uncorrelated noise}$$

Cross-spectrum: $f_{yx}(\omega) = e^{-2\pi i d\omega} f_x(\omega)$

- **Extension I**: **Multiply** and **delay**

$$y_t = a \cdot x_{t-d}$$

Cross-spectrum: $f_{yx}(\omega) = a \cdot e^{-2\pi i d\omega} f_x(\omega)$

- **Extension II**: **Linear filtering**:

$$y_t = \sum_{d=-\infty}^{\infty} a_d x_{t-d}, \qquad \sum_{t \in \mathbb{Z}} |a_t| < \infty,$$

Cross-spectrum:

$$f_{yx}(\omega) = \overbrace{\sum_{d=-\infty}^{\infty} a_d e^{-2\pi i d\omega}}^{A(\omega)} f_x(\omega) = A(\omega) f_x(\omega)$$

($A(\omega)$ is the **frequency response** of the filter).

Spectral Representation of a Vector Stationary Process

- Example 4.20 (and Property 4.18):
  Consider a **jointly stationary** bivariate process (x_t, y_t). The **autocovariance matrix** is

$$\Gamma(h) = \begin{pmatrix} \gamma_x(h) & \gamma_{xy}(h) \\ \gamma_{yx}(h) & \gamma_y(h) \end{pmatrix},$$

  and the **spectral matrix** is

$$\boldsymbol{f}(h) = \begin{pmatrix} f_x(h) & f_{xy}(h) \\ f_{yx}(h) & f_y(h) \end{pmatrix}.$$

- **Note**: Hermitian symmetry: $\Gamma^*(\omega) = \Gamma(\omega)$.
- Obvious extensions to **higher dimensions**.

Recap

- Periodogram indicates the **component of data variance explainable by sinusoids** at frequency j.
- The **spectral density** $f(\omega)$ has a **Fourier series** representation with coefficients given by the **covariance function** $\gamma(h)$.
- The **spectral density** gives **typical size** of random variable **periodogram**.

- The **cross-spectral density** $f_{xy}(\omega)$ has a **Fourier series** representation with coefficients given by the **cross covariance function** $\gamma_{xy}(h)$.

- The **spectral density** and **cross-spectral density** play nicely with linear filtering.

Next:

- Spectral **estimation**.
- Frequency domain **regression**.

Spectral Estimation

Properties of Periodogram (review)

- Let (w_t) be Gaussian white noise. n is odd. Then

$$I_n(\omega_{j:n}) \overset{iid}{\sim} \mathrm{Exp}(\sigma_w^2), \quad 1, \ldots, (n-1)/2.$$

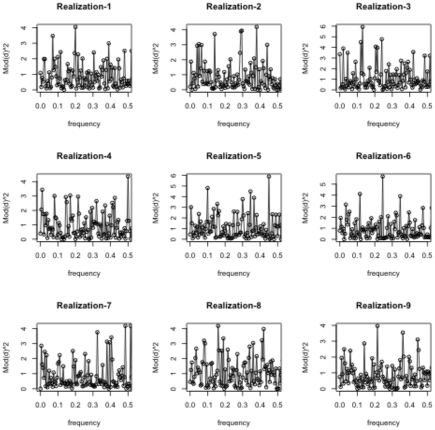
Alternately,

$$\frac{2I_n(j/n)}{\sigma_w^2} \overset{iid}{\sim} \chi_2^2, \quad 1, \ldots, (n-1)/2.$$

- Periodogram of a Gaussian White Noise is an "Exponential/Chi-squared Noise".

Periodogram of White Noise - Several examples

```
n = 200; freq = (0:(n-1))/n; par(mfrow=c(3,3))
for (i in 1:9) {
    d <- fft(rnorm(n)) / sqrt(n)
    plot(freq, Mod(d)^2, type='o', xlab='frequency',
         main = paste('Realization',i, sep='-'), xlim=(c(0,0.5)))
}
```



Smoothing the Periodogram, I

- $L = 2m + 1$ for integer $m > 0$, $L < n/2$.
- **Running average** periodogram **smoother**:

$$\bar{f}(j/n) \equiv \frac{1}{L} \sum_{k=-m}^{m} I_n\left(\frac{j+k}{n}\right), \quad j = 1, \ldots, (n-1)/2.$$

- Interpret indices **circularly**,

$$I_n(\pm k/n) = I_n\left((n \pm k)/n\right), \qquad k = 0, \pm 1, \pm 2, \ldots.$$

- **Remark:** In practice we use a **weighted** average rather than a **uniform** average.

Smoothing the Periodogram, II

- For n odd, set

$$\omega_{j:n} \equiv \frac{j}{n}, \qquad j = 1, \ldots, \frac{n-1}{2}.$$

- If (w_t) is Gaussian white noise,

$$\frac{2L\bar{f}(\omega_{j:n})}{\sigma_w^2} \sim \chi^2_{2L}, \quad j = 1, \ldots, (n-1)/2.$$

- Why?
  - Periogoram of a GWN is proportional to χ^2_2.
  - $L\bar{f}(\omega_{j:n})$ is proportional to the sum of L independent χ^2_2.
  - Sum of L independent χ^2_2 is χ^2_{2L}.

- Definition: Let $Z_1, \dots, Z_n \overset{iid}{\sim} (0,1)$. Then

$$\sum_{i=1}^{n} Z_i^2 \sim \chi_n^2$$

has a **Chi-squared distribution on n degrees of freedom**.

- Properties of χ_n^2:
  - Probability density function

$$f(x;k) = \begin{cases} \frac{x^{(k/2)-1}e^{-x/2}}{2^{k/2}\Gamma(k/2)}, & x \geq 0; \\ 0, & \text{otherwise.} \end{cases}$$

    ($\Gamma(k/2)$ is the Gamma function)
  - $\mathbb{E}\left[\chi_n^2\right] = n$.
  - $\chi_2^2 = \mathrm{Exp}(2)$.

Smoothing the Periodogram, III

- Property: Let (x_t) be Gaussian and stationary with

$$\sum_{h=-\infty}^{\infty} \sqrt{|h|}|\gamma(h)| < \infty.$$

Then

$$\frac{2L\bar{f}(\omega_{j:n})}{f_x(\omega_{j:n})} \overset{approx}{\sim} \chi^2_{2L}, \quad j = 1, \ldots, (n-1)/2.$$

- Why?
  - $I_n(\omega_{j:n})$ are approximately independent $f_x(\omega_{j:n})\chi^2_2/2$.
  - If $L \ll n$, $L\bar{f}(\omega_{j:n})$ is the **sum** of L approximately independent $f_x(\omega_{j:n})\chi^2_2/2$.

Spectral Density and Periodogram (review)

- Suppose (x_t) is **stationary** with **absolutely summable** $\gamma_x(h)$.

$$f_x(\omega) = \lim_{n\to\infty} \mathbb{E}\left[I_n\left(\lfloor n\omega \rfloor / n\right)\right], \qquad \omega \in (0, 1/2).$$

- Suppose that x_t is also a **Gaussian** process. Then, approximately

$$I_n(\omega_{j:n}) \overset{approx}{\sim} \mathrm{Exp}\left(f_x(\omega_{j:n})\right), \qquad j = 0, 1, \ldots, n/2.$$

  Equivalently (Property 4.6),

$$\frac{2 I_n(\omega_{j:n})}{f_x(\omega_{j:n})} \overset{approx}{\sim} \chi^2_2, , \qquad j = 0, 1, \ldots, n/2.$$

- **In words**: "Spectrum gives **typical size** of random variable **periodogram**."

Smoothing the Periodogram, IV

- Define

$$E_{j,n} \equiv \left\{ \frac{2L\bar{f}(\omega_{j:n})}{\chi^2_{2L}(1-\frac{\alpha}{2})} \le f_x(\omega_{j:n}) \le \frac{2L\bar{f}(\omega_{j:n})}{\chi^2_{2L}(\frac{\alpha}{2})} \right\}, \qquad \alpha > 0 \text{ is small.}$$

- From

$$\frac{2L\bar{f}(\omega_{j:n})}{f_x(\omega_{j:n})} \overset{approx}{\sim} \chi^2_{2L}, \quad j = 1, \ldots, (n-1)/2,$$

we have

$$\Pr(E_{j,n}) \to 1 - \alpha$$

"$E_{j,n}$ is an asymptotic $(1-\alpha)$ confidence interval for $f_x(\omega)$".

Smoothing the Periodogram, V

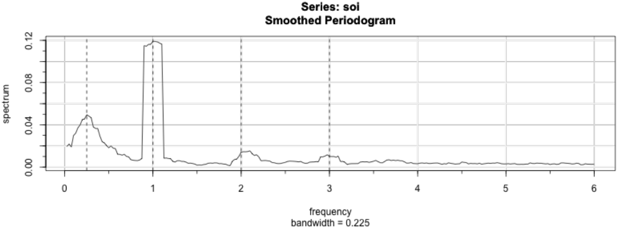
- Let (x_t) be stationary Gaussian. Set

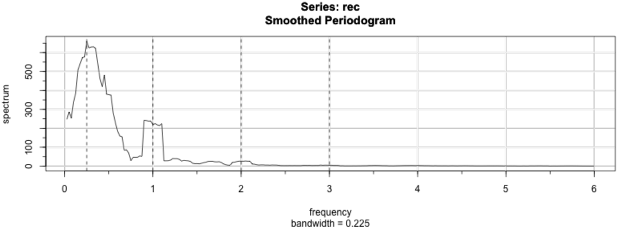
$$C_{j,n} \equiv \Big[\log\left(\bar{f}(\omega_{j:n})\right) + \log(2L) - \log\left(\chi^2_{2L}(1-\frac{\alpha}{2})\right),$$
$$\log\left(\bar{f}(\omega_{j:n})\right) + \log(2L)\log\left(\chi^2_{2L}(\frac{\alpha}{2})\right)\Big].$$

- $\Pr\left(\log(f_x(\omega_{j:n})) \in C_{j,n}\right) \to 1-\alpha$. "$C_{j,n}$ is an asymptotic $(1-\alpha)$ confidence interval for $\log(f_x(\omega_{j:n}))$."

Example 4.14: Smoothed Periodogram for SOI & Recruitment

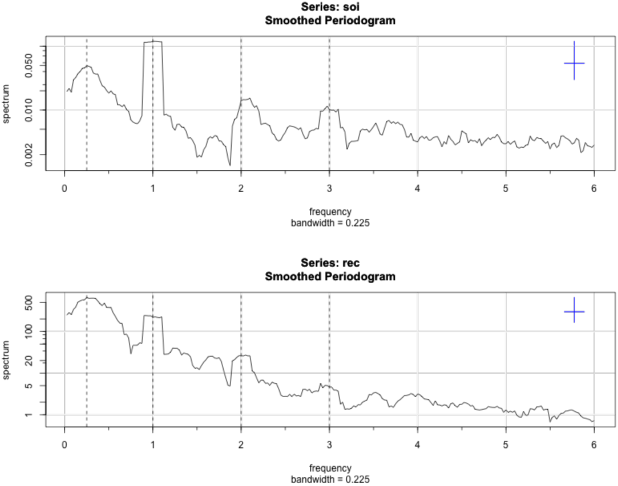
```
soi.ave = mvspec(soi, kernel('daniell',4), log='no')
abline(v=c(.25,1,2,3), lty=2)
soi.ave$bandwidth  # = 0.225
...
```





Log-Smoothed Periodogram for SOI & Recruitment

```
soi.ave = mvspec(soi, kernel('daniell',4), log='yes')
abline(v=c(.25,1,2,3), lty=2)
soi.ave$bandwidth  # = 0.225
...
```



Coherence Estimation, I

- Discrete Fourier Transform:

$$d_x(j/n) = \frac{1}{n}\sum_{t=1}^{n} x_t e^{-2\pi i j/n}, \qquad j = 1, \ldots, n-1.$$

$$d_y(j/n) = \frac{1}{n}\sum_{t=1}^{n} y_t e^{-2\pi i j/n}, \qquad j = 1, \ldots, n-1.$$

- Definition: **Cross-periodogram**

$$I_{yx}(\omega) = d_y(\omega) d_x^*(\omega)$$

Coherence Estimation, II

- **Running average cross-periodogram smoother** :

$$\bar{f}_{yx}(j/n) \equiv \frac{1}{L} \sum_{k=-m}^{m} I_{yx}\left(\frac{j+k}{n}\right), \quad j = 1, \ldots, (n-1)/2,$$

where

  - $L = 2m + 1$ for integer $m > 0$, $L < n/2$.
  - Interpret indices **circularly**, $I_{yx}(\pm k/n) = I_{yx}\left((n \pm k)/n\right)$.

- **Squared Coherence estimate** (uniform weights):

$$\bar{\rho}^2_{xy}(\omega_{j:n}) = \bar{\rho}^2_{yx}(\omega_{j:n}) = \frac{|\bar{f}_{xy}(\omega_{j:n})|^2}{\bar{f}_x(\omega_{j:n})\hat{f}_y(\omega_{j:n})}.$$

- Property: If $\rho_{yx}(\omega) = 0$,

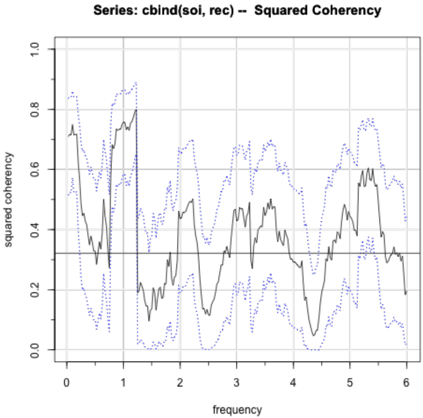
$$\frac{\bar{\rho}^2_{yx}(\omega)}{\left(1 - \bar{\rho}^2_{yx}(\omega)\right)}(L-1) \overset{approx}{\sim} F^2_{2L-1}.$$

Used for testing **against** the null: "no coherence at freq ω".

Warning: **multiple testing**; see discussion around Eq. 4.63.

Example 4.21 Squared Coherence SOI & Recruitment

```
sr = mvspec(cbind(soi,rec), kernel("daniell",9),
            plot.type="coh", plot=TRUE)
sr$df                      # df = 35.8625
f = qf(.999, 2, sr$df-2)   # f = 8.529792
C = f/(18+f)               # C = 0.3188779
abline(h = C)
```



The two series are strongly coherent at: Annual Cycle of 12mo, El-ninõ Cycle 3-7years (peak is at 9years).

Frequency-domain Regression

Lagged Regression Setting

- Lagged regression model

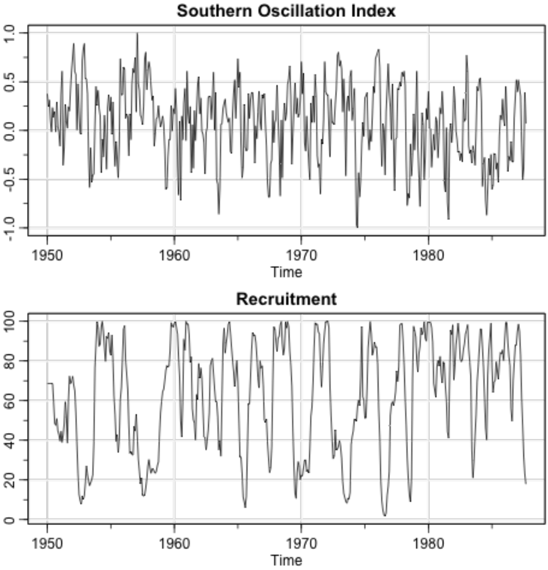
$$y_t = \sum_{r=-\infty}^{\infty} \beta_r x_{t-r} + v_t,$$

  where:

  - (v_t) is stationary **noise**,
  - (x_t) is the stationary **observed input** series (aka predictor or covariate)
  - (y_t) is the **observed output** series.

- **Goal**: Estimate filter coefficients (β_r).
- Example: SOI and Recruitment.
- In Lecture 9 we have used the "transfer function modelling" approach to this setting.

SOI & Recruitment



Normal Equations

- Write spectral matrix of (x_t, y_t):

$$\boldsymbol{f}(\omega) = \begin{pmatrix} f_x(\omega) & f_{xy}(\omega) \\ f_{yx}(\omega) & f_y(\omega) \end{pmatrix}$$

- **Assume** that (x_t) and (y_t) have **zero means**.
- The MSE

$$\mathrm{MSE}_t = \mathbb{E}\left[\left(y_t - \sum_{r=-\infty}^{\infty} \beta_r x_{t-r}\right)^2\right].$$

The **optimal** (β_r) satisfies the **orthogonality condition**

$$\mathbb{E}\left[\left(y_t - \sum_{r=-\infty}^{\infty} \beta_r x_{t-r}\right) x_{s+t}\right] = 0, \qquad \forall s = 0, \pm 1, \pm 2, \ldots.$$

- Corollary: The **Normal Equations**:

$$\sum_{r=-\infty}^{\infty} \beta_r \gamma_x(s-r) = \gamma_{yx}(s), \qquad s = 0, \pm 1, \pm 2, \ldots.$$

A Spectral Approach for Solving the Normal Equations

- The **Normal Equations**:

$$\sum_{r=-\infty}^{\infty} \beta_r \gamma_x(s-r) = \gamma_{yx}(s), \qquad s = 0, \pm 1, \pm 2, \ldots.$$

- Write the **spectral representation** of both sides:

$$\gamma_{yx}(s) = \int_{-\frac{1}{2}}^{\frac{1}{2}} f_{yx}(\omega) e^{2\pi i \omega s} d\omega,$$

and

$$\sum_{r=-\infty}^{\infty} \beta_r \gamma_x(s-r) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \sum_{r=-\infty}^{\infty} \beta_r f_x(\omega) e^{2\pi i \omega (s-r)} d\omega$$
$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} B(\omega) f_x(\omega) e^{2\pi i \omega s} d\omega,$$

$B(\omega)$ is the **frequency response** of the filter (β_r).

- Normal Equations in the **frequency domain**:

$$B(\omega) f_x(\omega) = f_{yx}(\omega), \quad \omega \in [-1/2, 1/2].$$

Prescription for Lagged Regression:

STEP I: The equation $B(\omega) f_x(\omega) = f_{yx}(\omega)$ suggests

$$\hat{B}(\omega_k) = \frac{\hat{f}_{xy}(\omega_k)}{\hat{f}_x(\omega_k)}$$

as the **estimator of the Fourier transform** of the regression coefficients, over a grid $\omega_k = k/M$ with $M \ll n$.

STEP II: Assuming that $B(\omega)$ is smooth, use:

$$\hat{\beta}_t = \frac{1}{M} \sum_{k=0}^{M-1} \hat{B}(\omega_k) e^{2\pi i \omega_k t}, \qquad t = 0, \pm 1, \pm 2, \ldots, \pm(M/2 - 1).$$

$$\hat{\beta}_t = 0, \qquad |t| \geq M/2.$$

- Straightforward extension to **vectorial** (x_t) (Section 7.2).

Example 4.24: SOI and Recruitment

- **High coherence** suggests a **lagged regression** relation.

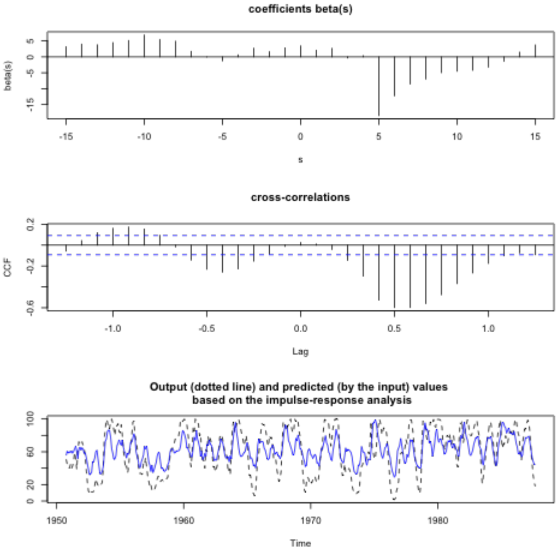
```
LagReg(soi, rec, L=15, M=32, inverse=TRUE, threshold=.01)
```

```
        INPUT: soi OUTPUT: rec   L = 15    M = 32

The coefficients beta(0), beta(1), beta(2) ... beta(M/2-1) are
3.463141 2.088613 2.688139 -0.3515829 0.3717705 -18.47931 -12.2633
-8.539368 -6.984553 -4.978238 -4.526358 ... 1.489903 3.744727

The positive lags, at which the coefficients are large
in absolute value, and the coefficients themselves, are:
     lag s    beta(s)
[1,]     5 -18.479306
[2,]     6 -12.263296
[3,]     7  -8.539368
[4,]     8  -6.984553

The prediction equation is
rec(t) = alpha + sum_s[ beta(s)*soi(t-s) ], where alpha = 65.96584
MSE = 414.0847
```

- Suggested model:

$$y_t = 66 - 18.5x_{t-5} - 12.3x_{t-6} - 8.5x_{t-7} - 7x_{t-8} + w_t.$$

SOI and Recruitment (cont'd)



Frequency-Domain Representation of the MSE

- **Normal Equations** in the **frequency domain**:

$$B(\omega) f_x(\omega) = f_{yx}(\omega), \quad \omega \in [-1/2, 1/2].$$

- **Minimized** MSE satisfies

$$\mathrm{MSE}_t = \mathbb{E}\left[\left(y_t - \sum_{r=-\infty}^{\infty} \beta_r x_{t-r}\right) y_t\right] = \gamma_y(0) - \sum_{r=-\infty}^{\infty} \beta_r \gamma_{xy}(-r),$$

which is independent of t.

- Implies the **frequency-domain** representation:

$$\mathrm{MSE} = \int_{-\frac{1}{2}}^{\frac{1}{2}} [f_y(\omega) - B(\omega) f_{xy}(\omega)]\, d\omega = \int_{-\frac{1}{2}}^{\frac{1}{2}} f_y(\omega) \left[1 - \rho_{yx}^2(\omega)\right] d\omega$$

(large coherence leads to small MSE).

Optimum Filtering (Wiener-Kolmogorov Problem)

- Model:

$$y_t = \sum_{r=-\infty}^{\infty} \beta_r x_{t-r} + v_t,$$

where

  - (v_t) is a **noise process** uncorrelated of (x_t).
  - (β_r) are **known**.
  - (y_t) is **observed**. (x_t) is **not observed**.

- **Goal**: Find an estimator for (x_t) of the form

$$\hat{x}_t = \sum_{r=-\infty}^{\infty} a_r y_{t-r}.$$

- Solved by Norbert Wiener and Andrey Kolmogorov in the 1940's.

Frequency-Domain Approach

- **Orthogonality principle**: optimum (a_r) satisfies

$$\mathbb{E}\left[\left(x_t - \sum_{r=-\infty}^{\infty} a_r y_{t-r}\right) y_{t-s}\right] = 0 \Rightarrow \sum_{r=-\infty}^{\infty} a_r \gamma_y(s-r) = \gamma_{xy}(s)$$

for all $s = 0, \pm 1, \pm 2, \ldots$.

- Use **spectral representation** and properties of **linear filters**:

$$A(\omega) f_y(\omega) = f_{xy}(\omega)$$

$$A(\omega)\left(|B(\omega)|^2 f_x(\omega) + f_v(\omega)\right) = B^*(\omega) f_x(\omega)$$

where: $A(\omega)$, $B(\omega)$ are the frequency responses of (a_r), (β_r).

- **Optimal** filter's frequency response:

$$A(\omega) = \frac{B^*(\omega)}{|B(\omega)|^2 + \mathrm{SNR}(\omega)}$$

- **Minimized** MSE

$$MSE^* = \int_{-\frac{1}{2}}^{\frac{1}{2}} \left[f_x(\omega) - \frac{|B(\omega)|^2}{(|B(\omega)|^2 + \mathrm{SNR}(\omega))^2} \right] d\omega$$