Equilibrium Inefficiency in Resource Buying Games with Load-Dependent Costs

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Abstract. We study the inefficiency of equilibria of *resource buying games*, i.e., congestion games with *arbitrary cost-sharing*. Under arbitrary cost-sharing, players do not only declare the resources they will use, they also declare and submit a payment per resource. If the total payments on a resource cover its cost, the resource is activated, otherwise it remains unavailable to the players. Equilibrium existence and inefficiency under arbitrary cost-sharing is very well understood in certain models, such as network design games, where the joint cost of every resource (edge) is constant. In the case of congestion-dependent costs the understanding is not yet complete. For increasing per player cost functions, it is known that the optimal solution can be cast as a Nash equilibrium with the appropriate selection of payments and, hence, the price of stability is 1. In this work we initially focus on the price of anarchy for linear congestion games and prove that (in the direct generalization of the arbitrary cost-sharing model to congestion-dependent costs) it grows to infinity as the number of players grows large. However, we also show that with a natural modification to the cost-sharing model, the price of anarchy becomes 17/3. Turning our attention to strong Nash equilibria, we show that the worst-case inefficiency of the best and worst stable outcomes remains the same as for Nash equilibria, with the strong price of stability staying at 1 and the strong price of anarchy staying at 17/3. These results imply arbitrary cost-sharing is comparable to fair cost-sharing as it has a better best-case scenario and a (slightly) worse worst-case scenario. We also study models with restricted strategy sets (uniform matroid congestion games) and properties of best response dynamics with arbitrary cost-sharing.

1 Introduction

The class of *unweighted congestion games* [30] includes a large collection of applications where players compete for the use of resources with congestion-dependent costs. Players are called to select the subsets of resources they will use, with each one of them having a *strategy set* of allowable such subset selections, and these decisions induce joint costs on the resources as dictated by their respective activation-cost functions. These joint costs are split among the users of resources in a way specified by the *cost-sharing policy* of the game. Players are expected to reach a stable outcome, such as a *pure Nash equilibrium* (NE), i.e., a solution robust against unilateral deviations, or a *strong Nash equilibrium* (SE), i.e., a solution robust against group deviations.[4] Metrics of interest from the perspective of the system designer include the *price of anarchy* (PoA), i.e., the worst case ratio of the total cost in a NE divided by the optimal cost, the *price of stability* (PoS), i.e., the worst case ratio of the total cost in the best NE divided by the optimal cost, and, similarly for SE, the *strong price of anarchy* (SPoA) and the *strong price of stability* (SPoS).

A large body of work studies the above setting under the *fair cost-sharing* policy which dictates that the joint cost of a resource is split equally among its users. Among the most fundamental classes of games in these studies one finds *network design games* where a player's strategy set consists of all possible paths in a graph between the player's designated endpoints and where the joint cost of every edge is a given constant, together with *linear congestion games*, where the joint cost of a resource is quadratic in the number of players using it (with the per-player cost being linear). For network design games, [6] shows that the PoA is equal to the number of players n, whereas the PoS is $\Theta(\log n)$. For linear congestion games, the PoA was shown to be $5/2$ in [9,16] and the PoS was shown to be $1 + \sqrt{3}/3$ in [11,15]. The SPoA was shown to also be $5/2$ in [14]. Generalizations of fair cost-sharing to weighted versions of congestion games are studied in [2,10,31].

Different kinds of cost-sharing policies for congestion games have also been studied. For example, [13,28] study various types of cost-sharing methods for network design games (such as the class of *weighted Shapley values*). In the congestion-dependent costs setting, which includes linear congestion games, [21] shows that fair cost-sharing minimizes the PoA among all cost-sharing policies that dictate player costs. Other literature that studies cost-sharing in congestion games and their weighted variants includes [19,22,34].

A different flavor of cost-sharing is given by *arbitrary cost-sharing*, which induces the class of *resource buying games*. In contrast to the methods described above, which prescribe player costs on a resource, arbitrary cost-sharing allows players to declare their cost shares. Specifically, each player picks the resources that he will use and submits a different payment for each one. If the total payments for a given resource cover the cost induced by its users, the resource is activated. In the opposite case, the resource remains inaccessible. This setting has been studied comprehensively for network design games. The work in [7] shows that a NE is not guaranteed to exist under arbitrary cost-sharing and that the PoA and PoS are large (almost equal to the number of players n). For the special case of a common destination node, the PoS is 1 and a NE is guaranteed to exist. An SPoA of $\Theta(\log n)$ is given in [17]. Other works that study arbitrary cost-sharing in network design games include [4,5,8,12,24,26]. Summarizing the results and comparing against fair cost-sharing, we observe that, in the general network design game, arbitrary costs-sharing loses the NE existence property

[4] In this paper, we consider *pure* strategies, as is common in the study of resource buying games.

and increases the PoS from logarithmic to linear. The situation improves for the common destination case where NE existence is maintained and the PoS improves to 1. Interestingly, a special case of network design games where even the PoA improves has been identified in the face of *real-time scheduling games* [33] in which the PoA drops from $\Theta(\sqrt{n})$ for fair cost-sharing to 2 for arbitrary cost-sharing [20].

Less is known about resource buying games with congestion-dependent costs. The work in [25] studies classes of games with non-decreasing per player costs. Most closely related to our setting is the work in [23], which shows that for increasing per player costs, a NE always exists and that, in fact, the optimal solution can be made to be a NE with appropriate payments, thus settling the PoS to be equal to 1. In this landscape, our work sets out to further investigate the inefficiency of equilibria in linear congestion games and compare against fair cost-sharing, which achieves PoA and SPoA 5/2, and PoS $1 + \sqrt{3}/3$.

Other related work deals with selfish and greedy load balancing. In natural dynamics, a player that joins a resource needs to cover the marginal change in the resource activation cost. This property also characterizes selfish load balancing instances [32,11]. Some of our results for matroid games generalize results from these papers.

1.1 Our Results

We initially study the obvious generalization of arbitrary cost-sharing to linear costs, in which players can submit any payment for a resource. We quickly observe that the PoA can be very large with a simple and somewhat uninteresting example, the details of which are given in Section 3. The example relies on having one player who is restricted to a single resource and multiple others who freeload on him instead of switching to empty resources. The restricted player ends up paying an astronomical cost of n^2 even though his marginal contribution to the joint cost is much smaller (specifically $2n - 1$). Given that such an instance is unreasonable from a practical perspective (a player would not tolerate paying a very large part of a resource's cost that is clearly not caused by his presence so that others may use it), we seek a minor modification to the arbitrary cost-sharing model that leads to more meaningful results. We choose to impose the *marginal contribution constraint*, which suggests that no payment larger than the marginal contribution of a player on a resource is accepted.

The marginal contribution constraint can be interpreted in two ways. In the first one, the system designer closes down resources where the constraint is violated. This is a means for the designer to reduce the PoA in a manner that is instance-oblivious, i.e., requires only local observation of the players and payments on each resource as opposed to global knowledge of the full set of players and their available strategies. In the second interpretation, players suffer a large cost when they pay more than their marginal contribution due to the perception of being exploited and they themselves deviate away from such strategies.

Some of our results refer to uniform matroid resource buying game, in which every player j is associated with a set of feasible resources, and a demand ℓ_j.

The strategy space of a player includes all the subsets of size ℓ_j of his feasible resources. A singleton game is a special case of matroid game with unit demands. A prominent example of uniform matroid games, is preemptive real-time scheduling, where every player corresponds to a job of of a specific length that should be processed after its release- time and before its deadline. Since preemptions are allowed, any selection of ℓ_j slots in this interval can do.

Our results on arbitrary cost-sharing with the marginal contribution constraint in linear congestion games are as follows:

– In Section 3 we prove that the PoA and SPoA for general games are equal to 17/3 and the SPoS is equal to 1. We also show that a NE always exists.
– In Section 4 we prove that the PoA and SPoA for the special case of uniform matroid games reduces to a value between 4 and 4.055. We also show that in a singleton game, the minimal size of a coalition that may benefit from a coordinated deviation from a NE profile is 3, thus a NE is stable against any coordination of two players. We also show that while the worst-case PoA is equal to the worst-case PoA, there are games for which the PoA is higher than the SPoA.
– In Section 5 we discuss convergence properties of best-response dynamics, showing that convergence is typically faster than fair cost-sharing. For uniform matroid games we suggest a rule for selecting the deviating player in every BR step, such that convergence is guaranteed in time lower than the players' total demand.

2 Model

In the *linear resource buying games* that we study, there is a set of n unweighted players N and a set of m resources E. Each player $j \in N$ selects a set $p_j \subseteq E$ of resources that he will use, from a set of available such profiles $S_j \subseteq 2^E$.

A profile p_j, together with payments $\xi_{e,j}$ for each $e \in p_j$ constitute the strategy (p_j, ξ_j) of player j. We write p for the complete profile and $f_e(p)$ for the load on e in p, that is, the number of players using e in p. Every resource e induces an activation cost $c_e(f_e(p)) = f_e(p)^2$ (by convention, games with such costs are called *linear*, given that the *per player* cost on a resource is linear). The players have to cover this cost with their payments. We write (p, ξ) for the complete strategies of all players. Each player j seeks to minimize his cost which is:

$$cost_j(p, \xi) = \begin{cases} \sum_{e \in p_j} \xi_{e,j}, & \text{if all } e \in p_j \text{ are open} \\ +\infty, & \text{otherwise.} \end{cases} \quad (1)$$

A resource is open if its activation cost is paid for by the players, i.e., when:

$$\sum_{j: e \in p_j} \xi_{e,j} \geq c_e(f_e(p)).$$

Given the cost structure defined above, we may describe the solution concepts we study in this paper, namely the *pure Nash equilibrium* (NE) and the *strong*

Nash equilibrium (SE). The NE condition enforces that no player should be able to unilaterally change his declared payments and/or set of resources and reduce his cost. Formally, in a NE (p, ξ), we have that for every player j and every strategy (p'_j, ξ'_j) of that player:

$$cost_j(p, \xi) \le cost_j(\{p_{-j}, p'_j\}, \{\xi_{-j}, \xi'_j\}).$$

The SE condition enforces that there should not be a set of players Γ who can coordinate to change their strategies in a way such that every one of them reduces his cost. Formally, for a SE (p, ξ) we have that, for every subset of players Γ, and for every collection of strategies (p'_Γ, ξ'_Γ) of these players, there exists some player $j \in \Gamma$ such that:

$$cost_j(p, \xi) \le cost_j(\{p_{-\Gamma}, p'_\Gamma\}, \{\xi_{-\Gamma}, \xi'_\Gamma\}).$$

Best-response dynamics (BRD) is a natural method by which players proceed toward a pure Nash equilibrium via a local search method. Player j is said to be *sub-optimal* in (p, ξ) if he can reduce his cost by a unilateral deviation, i.e., if there exists (p'_j, ξ'_j) such that

$$cost_j(\{p_{-j}, p'_j\}, \{\xi_{-j}, \xi'_j\}) < cost_j(p, \xi).$$

In BRD, as long as the strategy profile is not a NE, a sup-optimal player is chosen to deviate to a strategy that will minimize his cost, given the profile of others.

Some of our results refer to *uniform matroid games* in which every player j is associated with a subset $M_j \subseteq E$ of the resources, and a demand ℓ_j. The strategy space of player j includes all subsets of M_j of size ℓ_j. A *singleton* game is a special case in which $\forall j, \ell_j = 1$.

The cost of a profile (p, ξ) is the total players' cost, that is, $cost(p, \xi) = \sum_j cost_j(p, \xi)$. We denote by $OPT(G)$ the cost of a social optimal solution of a game G.

We conclude the section by defining our performance metrics. We quantify the inefficiency incurred due to self-interested behavior according to the *price of anarchy* (PoA) [29] and *price of stability* (PoS) [6] measures. The PoA is the worst-case inefficiency of a pure Nash equilibrium, while the PoS measures the best-case inefficiency of a pure Nash equilibrium. Formally, Let $\mathcal{G}$ be a family of games, and let G be a game in $\mathcal{G}$. Let $\Upsilon(G)$ be the set of pure Nash equilibria of the game G. Assume that $\Upsilon(G) \neq \emptyset$.

- The *price of anarchy* of G is the ratio between the *maximal* cost of a NE and the social optimum of G. That is, $\text{PoA}(G) = \max_{p \in \Upsilon(G)} cost(p)/OPT(G)$. The *price of anarchy* of the family of games $\mathcal{G}$ is $\text{PoA}(\mathcal{G}) = sup_{G \in \mathcal{G}} \text{PoA}(G)$.
- The *price of stability* of G is the ratio between the *minimal* cost of a NE and the social optimum of G. That is, $\text{PoS}(G) = \min_{p \in \Upsilon(G)} cost(p)/OPT(G)$. The *price of stability* of the family of games $\mathcal{G}$ is $\text{PoS}(\mathcal{G}) = sup_{G \in \mathcal{G}} \text{PoS}(G)$.

The *strong price of anarchy* (SPoA) and the *strong price of stability* (SPoS) introduced in [3] are defined similarly, where $\Upsilon(G)$ refers to the set of strong equilibria.

3 General Resource Buying Games

We begin by proving that the PoA of resource buying games with linear per player costs (i.e., resource activation costs $c(x) = x^2$) grows to infinity with the number of players.

Theorem 1. *The PoA of linear resource buying games is $\Omega(n)$.*

Proof. Consider a game with n players and n resources. Player 1 can only pick resource 1, i.e., $S_1 = \{\{1\}\}$. Every other player j can pick either resource 1 or resource j, i.e., $S_j = \{\{1\}, \{j\}\}$. Let p be the profile in which every player picks resource 1 and let the declared payments be $\xi_{1,1} = n^2 - (n-1)$ and $\xi_{1,j} = 1$ for every $j > 1$.

We observe that (p, ξ) is a NE as follows. The players receive service on resource 1, since they have covered its cost. Hence, no one has an incentive to increase the payment there. Decreasing the payment will result in losing service, hence there is no incentive for that either. Each player $j > 1$ also has the option to move to the alternative resource j. There j would have to pay 1, which offers no improvement in cost.

The cost of profile p is n^2. If we let p^* be the assignment in which every player j picks resource j, we get a cost of n. This proves the PoA is at least n. □

We observe that the high PoA is given by an unrealistic and uninteresting instance. It assumes that there is one player who will effectively suffer a very large cost so others can freeload on him. To correct for such degenerate outcomes, we impose the *marginal contribution constraint* which enforces that no player may declare a cost higher than $c_e(f_e(p)) - c_e(f_e(p) - 1)$, otherwise the resource remains unavailable. Note that this expression is the highest increase that the player can cause to the joint resource cost in any ordering of the resource's users and observe that such a constraint is implicit in the large literature of arbitrary cost-sharing in network design games: when an edge has unit cost, the largest increase a player can cause to the joint cost is 1 and that is precisely that max payment seen in a NE.

Through the rest of the paper and for simplicity of exposition, when analyzing equilibria we only consider outcomes in which all players are serviced and have a finite cost, i.e., outcomes on which the payments on every used resource equal its cost. While this is automatically true for the class of SE, some extra care needs to be taken to ensure it is also true for the class of NE or otherwise players can get stuck in low payment outcomes, e.g., when every player on a resource declares a 0 payment and unilateral increases cannot cover the resource cost without violating the marginal contribution constraint. We note that imposing a cost structure that addresses this is easy to achieve with a tweak on handling underpaid resources: Resources remain closed when a player is paying more than his marginal contribution but, when players underpay, each one is charged his bid plus twice the unpaid amount. Then each player has an incentive to increase his payment up to the marginal contribution until the resource costs are covered.

Hence, w.l.o.g., we may consider only outcomes in which all players are serviced. We next present our results on the inefficiency of equilibria of arbitrary cost-sharing with the marginal contribution constraint.

Let $\mathcal{G}$ be the class of linear resource buying games with the marginal contribution constraint.

Theorem 2. *$SPoS(\mathcal{G}) = 1$ (and hence also $PoS(\mathcal{G}) = 1$) and a pure NE exists for every $G \in \mathcal{G}$.*

Proof. Let p^* be an optimal profile. Assume that the players are ordered arbitrarily and every player is added greedily to his strategy in p^* and pays the marginal cost. By Theorem 6.1 in [23] this payment scheme produces a NE. We show it is also a strong NE. Assume by contradiction that p^* is not a SE and let Γ be a coalition. Let p' be the profile after the deviation of Γ. Let E^+, E^- denote the set of resources whose load increases and decreases respectively in the deviation of Γ, and let Δ_e denote the corresponding gap in the load on e.

$$\sum_e f_e(p')^2 - \sum_e f_e(p^*)^2 = \sum_{e \in E^+} ((f_e^* + \Delta_e)^2 - (f_e^*)^2) - \sum_{e \in E^-} ((f_e^*)^2 - (f_e^* - \Delta_e)^2) < 0.$$

To see why the last expression is negative, note that the first term is exactly the added cost on E^+ that the coalition Γ has to cover and the second term is the saved cost on E^-, which is at most what is saved by the coalition. Then, the fact that the expression is negative follows from the fact that the total cost of the coalition members strictly decreases. Hence, we get a contradiction to the optimality of p^*. □

We note that the above theorem easily generalizes to cost functions of the form $c(x) = x^d$ for $d > 1$. Now that we have shown that the nice properties of arbitrary cost-sharing from [23] still hold after our modification, we proceed to analyze the PoA and SPoA. We begin with a technical lemma that captures the well known *PoA smoothness* framework [31] in our model.

Lemma 1. *Suppose λ and $\mu < 1$ are positive real numbers such that for all integers $y \geq 1$ and $x \geq 0$ it holds that*

$$(2x + 1)y \leq \lambda y^2 + \mu x^2.$$

Then we get that the PoA of linear resource buying games with the marginal contribution constraint is at most $\lambda/(1 - \mu)$.

Proof. Let p_j^* be the set of resources used by player j in the optimal solution and let p_j be the set of resources used by player j in a worst case NE. Then, if $\xi_{e,j}$ is the payment of j for resource e, we get $\sum_j \sum_{e \in p_j} \xi_{e,j}$ for the total cost. Now consider the possible deviation of each player j, in which he uses the resources in p_j^* and pays:

- $\xi_{e,j}$ for each resource e that is both in p_j and p_j^*,
- $(f_e(p)+1)^2 - f_e(p)^2$ for each resource e that is in p_j^* but not in p_j (where $f_e(p)$ is the number of players on e in the NE p).

Note that this is a valid deviation from the NE, to the set of resources used by j in the optimal solution, since all resources in p_j^* will be paid for. By the equilibrium condition, each such cost is at least $\sum_{e\in E} \xi_{e,j}$, so we get:

$$\sum_{e\in E} f_e(p)^2 = \sum_j \sum_{e\in p_j} \xi_{e,j} \le \sum_j \sum_{e\in p_j^*\cap p_j} \xi_{e,j} + \sum_{e\in p_j^*\setminus p_j} (f_e(p)+1)^2 - f_e(p)^2$$
$$\le \sum_j \sum_{e\in p_j^*} (f_e(p)+1)^2 - f_e(p)^2 = \sum_j \sum_{e\in p_j^*} 2f_e(p)+1.$$

Here the last inequality follows by our marginal contribution constraint. Now if we let f_e^* be the number of players using resource e in the optimal solution and C^* be the optimal cost, we get:

$$\sum_{e\in E} f_e(p)^2 \le \sum_j \sum_{e\in p_j^*} 2f_e(p)+1 \le \sum_e \sum_{j:e\in p_j^*} 2f_e(p)+1 \le \sum_e (2f_e(p)+1)f_e(p^*)$$
$$\le \sum_e \lambda f_e(p^*)^2 + \mu f_e(p)^2 = \lambda \sum_{e\in E} f_e(p)^2 + \mu \sum_{e\in E} f_e(p^*)^2.$$

The last inequality follows by the assumption in the statement of the lemma. Rearranging gives

$$\frac{\sum_{e\in E} f_e(p)^2}{\sum_{e\in E} f_e(p^*)^2} \le \frac{\lambda}{1-\mu},$$

which proves the lemma.

Lemma 2. *$PoA(\mathcal{G}) \le 17/3$.*

Proof. Here we simply prove that values $\lambda = 3.4$ and $\mu = 0.4$ satisfy Lemma 1. Hence we focus on inequality:

$$(2x+1)y \le 3.4y^2 + 0.4x^2.$$

The inequality trivially holds for $y = 0$. We now focus on the case with $y = 1$. It is easy to check that the inequality holds for $x = 0, 1, 2, 3$. It is similarly easy to check that is holds for every real $x > 3$ since the derivative of $(0.4x^2 - 2x + 2.4)' = 0.8x - 2$ is positive for $x > 3$ and hence $0.4x^2 - (2x+1) + 3.4$ remains positive after $x = 3$.

We now switch to $y \ge 2$. Our main inequality can be rewritten as:

$$3.4y^2 + 0.4x^2 - 2xy - y \ge 0.$$

The value of x that minimizes the left hand side is $2.5y$. It is enough to satisfy the inequality with this value of x, which is:

$$3.4y^2 + 0.4 \cdot 2.5^2 y^2 - 5y^2 - y \ge 0 \Rightarrow 0.9y^2 - y \ge 0,$$

which is true since we have assumed $y \ge 2$.

□

Lemma 3. *$PoA(\mathcal{G}) \geq 17/3$.*

Proof. We construct the following instance with 7 players and 21 resources. The players are numbered $1, 2, \ldots, 7$ and the resources are labeled A_1, B_1, C_1, $\ldots$, A_7, B_7, C_7. Each player j wants to use one of two possible sets of resources. The first one, which will be the one used by the player in the optimal solution, is $\{A_j,\ B_j,\ C_j\}$. The second one, which will used by the player in the NE, is $\{A_{j+1}, A_{j+2}, A_{j+3}, B_{j+1}, B_{j+2}, C_{j+1}, C_{j+2}\}$. When the indices overflow (by becoming larger than 7) we assume we go back to 1 for 8, back to 2 for 9, and back to 3 for 10. In our NE we assume the players equally split the cost on every resource. Observe that every type A resource will have 3 players on it in the NE, while every type B and type C resource will have two players. Since each player uses 3 As, 2 Bs, and 2 Cs, each player's cost is: $3 \cdot 3 + 2 \cdot 2 + 2 \cdot 2 = 17$.

If a player j wishes to move to his other possible set of resources, he will have to cover the marginal increase to the costs there. He will be increasing the cost on resource A_j from 9 to 16, the cost on resource B_j from 4 to 9, and similarly the cost on resource C_j from 4 to 9. These give a total marginal payment of 17, which proves our assignment is indeed a NE. It is not hard to check that the total cost in the NE is 119 whereas the total cost in the optimal solution is 21. Taking the ratio completes the proof. □

Theorem 3. *$PoA(\mathcal{G}) = 17/3$.*

Proof. Follows from Lemma 2 and Lemma 3. □

Lemma 4. *For every $\epsilon > 0$, there exists a game $G \in \mathcal{G}$, such that $SPoA(G) \geq 17/3 - \epsilon$.*

Proof. We construct a game G, with n players and $3(n+3) + 18$ resources as follows. The players are numbered $1, 2, \ldots, n$, the first $3(n+3)$ resources are labeled $A_1, B_1, C_1, A_2, B_2,$
$C_2, \ldots, A_{n+3}, B_{n+3}, C_{n+3}$, and the final 18 resources are labeled $D_1, D_2, \ldots, D_{18}$. Each player j has two strategies:

- The first one, which will be used by the player in the optimal solution, is $p^*_j = \{A_j, B_j, C_j\}$, for $j \in \{4, 5, \ldots, n\}$, $p^*_1 = \{A_1, B_1, C_1, D_1, D_2, \ldots, D_9\}$, $p^*_2 = \{A_2, B_2, C_2, D_{10}, D_{11}, \ldots, D_{16}\}$, and $p^*_3 = \{A_3, B_3, C_3, D_{17}, D_{18}\}$.
- The second one, $p'_j = \{A_{j+1}, A_{j+2}, A_{j+3}, B_{j+1}, B_{j+2}, C_{j+1}, C_{j+2}\}$, will be used by the player in the SE.

Consider the profile (p', ξ) in which every player selects his second strategy, and the players equally split the cost on every resource. We first show that p' is a NE by examining each player separately:

- Player 1 is alone in $\{A_2, B_2, C_2\}$, shares $\{A_3, B_3, C_3\}$ with player 2, and shares $\{A_4\}$ with players 2 and 3. So $cost_1(p', \xi) = 1 \cdot 3 + 2 \cdot 3 + 3 \cdot 1 = 12$. If he switches to p^*_1, he would also be paying 12, since he would be alone on all 12 resources.

- Player 2 shares $\{A_3, B_3, C_3\}$ with player 1, two other A-resources with two other players, one B-resource with one other player and one C-resource with one other player. So $cost_2(p', \xi) = 2\cdot3+3\cdot2+2\cdot1+2\cdot1 = 16$. If he switches to p_2^*, he would be paying $(2^2-1)\cdot 3+1\cdot 7 = 16$.
- Player 3 shares three A-resources with two other players, two B-resources and two C-resources with one other player. So $cost_3(p', \xi) = 3\cdot3+2\cdot2+2\cdot2 = 17$. If he switches to p_3^*, he would be paying $(3^2-2^2)\cdot 3+2 = 17$.
- A player $j \in \{4, 5, \dots, n-2\}$ shares three A-resources with two other players, two B-resources and two C-resources with one other player. So $cost_j(p', \xi) = 3\cdot3+2\cdot2+2\cdot2 = 17$. A player that switches to p_j^*, he would be paying $(4^2-3^2)+(3^2-2^2)\cdot 2 = 17$.
- Player $n-1$ shares two A-resources with two other players, one A-resource, two B-resources and two C-resources with one other player. So $cost_{n-1}(p', \xi) = 3\cdot2+2\cdot1+2\cdot2+2\cdot2 = 16$. If he switches to p_{n-1}^*, he would be paying $(4^2-3^2)+(3^2-2^2)\cdot 2 = 17$.
- Player n shares one A-resource with two other players, one A-resource, one B-resource and one C-resource with one other player, and is alone on one A-resource, one B-resource and one C-resource. So $cost_n(p', \xi) = 3\cdot1+2\cdot1+2\cdot1+2\cdot1+1\cdot1+1\cdot1+1\cdot1 = 12$. If he switches to p_n^*, he would be paying $(4^2-3^2)+(3^2-2^2)\cdot 2 = 17$.

To see that (p', ξ) is a SE, suppose that an arbitrary subset of the players switch to their other strategy. Then the lowest-numbered player j in the subset experiences no improvement, since the resources that j would occupy if he switches are still occupied by the same players as in p'.

From the latter and from the fact that no resources are being shared in the optimal solution, we get that

$$SPoA(G) \geq \frac{17n-12}{3n+18} \to \frac{17}{3}, \text{ as } n \to \infty$$

This means that for every $\epsilon > 0$, the exists a game such that $SPoA(G) \geq 17/3 - \epsilon$, which completes the proof. □

Theorem 4. $SPoA(\mathcal{G}) = 17/3$.

Proof. Follows from Lemma 2 and Lemma 4. □

4 Uniform Matroid Resource Buying Games

In a uniform matroid resource buying game, every player j is associated with a subset $M_j \subseteq E$ of the resources, and a demand ℓ_j. The strategy space of player j includes all subsets of M_j of size ℓ_j. A singleton game is a special case of matroid games in which $\forall j, \ell_j = 1$. Let $\mathcal{G}_{UM}$ be the class of resource buying games with the marginal contribution constraint, and uniform matroid strategies.

For any resource buying game instance, a possible algorithm for computing a NE is to index the players arbitrarily, and then assign them in that order. If a player is added to a resource e with current load f_e, then $\xi_{e,j} = 2f_e + 1$. Every player selects a strategy that minimizes his total payment. It is easy to verify that the resulting profile is a NE, as the load on the resources can only increase after a player j is assigned, and every profitable deviation of j contradicts his greedy choice at the assignment time.

By the above, for the case of singleton games, we conclude that the PoA of our game is at least the approximation ratio of greedy load balancing with the objective of minimizing the loads' L_2-norm. This problem is studied in [32,11]. In fact, the lower bound of $4-\epsilon$ presented in [11], can be adapted for our game. We present a simpler lower bound that exploits the payment distribution flexibility in our game, and also handles coordinated deviations.

Theorem 5. *For every $\epsilon > 0$, there exists a game $G \in \mathcal{G}_{UM}$, with $SPoA(G) \geq 4 - \epsilon$. The lower bound is achieved already by a singleton game.*

We defer the proof to our full version. For the upper bound, we show that the elegant analysis in [11] for bounding the approximation ratio of greedy load balancing, can also be used for our game. The main challenge is to show that the 2-neighborhood property they define for the load balancing problem, is also valid in games with arbitrary payment distribution and uniform matroid strategies.

Theorem 6. $PoA(\mathcal{G}_{UM}) \leq 2\sqrt{21}/3 + 1 \approx 4.055$.

Proof. Let G be a uniform matroid game achieving the highest PoA, let (p, ξ) be a NE of G, and let p^* be an optimal solution such that the ratio of the total cost in p to the total cost in p^* is maximal. As shown in the proof of Lemma 2, for every profile (p, ξ) where ξ obeys the marginal contribution constraint, the total cost of p is at most $\sum_e (2f_e+1) f^*_e$. In addition, in every matroid game, the total load on the resources is fixed. Specifically, $\sum_{e \in E} f_e = \sum_{e \in E} f^*_e = \sum_j \ell_j$.

We can assume that the strategy space of player j is exactly $p_j \cup p^*_j$. That is, player j needs to select ℓ_j resources from $p_j \cup p^*_j$. If the set of feasible resources for j includes more resources, then they can be removed without hurting the stability of p.

Define a directed graph Δ as follows. The vertex set of Δ consists of one vertex for every resource. The edge set reflects the difference between p and p^* and is defined in the following way. For every player j, since G is a matroid game, $|p_j| = |p^*_j|$. Define a mapping $H_j : p^*_j \to p_j$. If $e \in p_j \cap p^*_j$ then $H_j(e) = e$, else, the mapping is arbitrary as long as it is 1-to-1 and onto. Player j contributes ℓ_j edges to Δ, one edge for every pair $(e, H_j(e))$. Thus, a directed edge may be a self loop (e, e) if player j uses e in both p and p^*, or an edge (e_1, e_2) if j uses resource e_1 only in the optimal solution, and resource e_2 only in the NE. We say that resource e is of type f_e/f^*_e. Note that f_e and f^*_e correspond, respectively, to the in-degree and out-degree of e in the graph Δ. We show that for any game instance we can construct another game instance that has at least the same PoA and satisfies the following 2-neighborhood property, defined in [11]: *the incoming edge of any resource of type* $1/1$ *originates from a resource of type* $0/1$. Formally (an extension of the definition in [11]), *Claim.* Let j be a resource for which $f_e = f_e^* = 1$. Assume $e \in p_a$ and $e \in p_b^*$. That is, player a is the only player using e in p, and player b is the only player using e in p^*. Then, (i) $a \neq b$, (ii) Let e' be the resource such that $H_a(e) = e'$, then $f_{e'} = 0$ and $f_{e'}^* = 1$.

Proof. (i) Assume by contradiction that $a = b$, that is, the same player is the only player that uses e in both profiles. Construct a new game instance by excluding resource e and reducing by one the demand of player a. If $\ell_j = 1$ in G, then player a is totally excluded from G. In the resulting game, both the optimal cost and the cost of p are decreased by 1, and, therefore, the PoA increases.

(ii) Given that $a \neq b$, the two players define a path $\langle e' - e - e'' \rangle$ in Δ, such that $H_a(e') = e$ and $H_b(e) = e''$. That is, $e' \in p_a^*, e \in p_a, e \in p_b^*$ and $e'' \in p_b$. Assume by contradiction that $f_{e'} > 0$, that is, e' is not empty in p. Construct a new game instance by (i) excluding resource e and reducing by one ℓ_a and ℓ_b. If a demand is reduced to 0, then exclude the corresponding player from G, (ii) introducing a new player c whose demand is $\ell_c = 1$ and whose strategy space is $\{e', e''\}$. Set $f_c^* = \{e'\}$ and $f_c = \{e''\}$. Also, set $\xi_{e'',c} = \xi_{e'',b}$, that is, the payment of the new player for using e'' is exactly the payment of b for using e''. We show that the resulting game has a higher PoA, by showing that the resulting profile is a NE. Since p is a NE, we know that b cannot benefit from replacing e'' by e. Since $f_e = 1$, this implies that $\xi_{e'',b} \leq 3$. In the new instance, the cost of c for using e'' is therefore at most 3. Our assumption that $f_{e'} > 0$ implies that replacing e'' by e' would result in cost at least 3 for c, thus, it is not beneficial, and the strategy of c is stable. No other player can benefit from changing his strategy, since all the loads are as in p. In the modified game, both the optimal cost and the cost of p are decreased by 1, and therefore the PoA increases.

We turn to show that $f_{e'}^* = 1$, that is, a is the only player that uses e' in p^*. Assume by contradiction that $f_{e'}^* > 1$. Thus, some other player, c, is together with a on e' in p^*. Construct a new game by introducing a new resource that is only feasible for a. In an optimal solution of the modified instance, a is the only player on the new resource, thus, the optimal cost is reduced by at least 3. On the other hand, p remains a NE, as also in p, a is using a resource with load 1. Again, we get a modified game with an increased PoA.

Summing up, the following three conditions hold: $\sum_e f_e^2 \leq \sum_e (2f_e + 1) f_e^*$, $\sum_{e \in E} f_e = \sum_{e \in E} f_e^*$, and the 2-neighborhood property is valid. Therefore, we have the three building blocks required for the analysis of [11] to get the PoA bound. □

In our full version, we also prove the following results on coordinated deviations in uniform matroid games.

Theorem 7. *The minimal size of a coalition that has a profitable deviation from a NE profile of a singleton game is* 3.

Theorem 8. *There exists a symmetric singleton game G and a NE profile (p, ξ) such that (p, ξ) is a NE, and there exists a set of* 3 *players that have a profitable coordinated deviation from (p, ξ).*

5 Convergence Rate of BRD

Given a strategy profile, the best-response (BR) of player j is the set of strategies that minimize his cost after fixing the strategies and payments of all other players. A player is sub-optimal in (p, ξ) if his current strategy is not in his BR set. If no player is sub-optimal, then (p, ξ) is a NE.

We analyze the convergence time of BRD by letting a player deviate to his BR and updating the payments of resources that the player departs from. We assume that these updates are not counted as a change of strategy and that only a change in the set of resources selected by a player counts as such. This fits analysis of BR convergence in other models – in which players costs are modified when other players change strategies.

It is well known that BRD converges to a NE in congestion games with fair cost-sharing. However, the BR-sequence may be exponentially long [18,1]. We first bound the number of steps in every BR sequence in a general resource buying game. The bound we achieve is identical to the bound for singleton games with fair cost-sharing [27].

Theorem 9. *For every resource buying game, and every initial profile (p_0, ξ_0), every BRD staring from (p_0, ξ_0) converges to a NE within less than $n^2 m$ steps.*

We defer the proof to our full version. For uniform matroid games, we suggest a rule for selecting in every BR step the deviating sub-optimal player, such that, if the initial profile is based on fair cost-sharing, then BRD converges within less than $\sum_j \ell_j$ steps. In particular, for singleton games, we get a bound of less than n steps on the convergence time, starting from an arbitrary profile with fair cost-sharing.

The intuition is that, unlike regular congestion games, the payment of a player does not increase if other players join resources he is using. Thus, every migration sets an upper bound on the cost of a player in the final NE.

Consider any BR sequence performed in a uniform matroid game. Denote by (p^t, ξ^t) the profile after t BR steps. In particular (p^0, ξ^0) is the initial profile. Observe that in a BR move of player j, he exchanges $k \leq \ell_j$ resources. Without loss of generality, every exchange is associated with a reduced cost. That is,

$$\text{For all } e_{out} \in p_j^t \setminus p_j^{t+1} \text{ and } e_{in} \in p_j^{t+1} \setminus p_j^t, \text{ it holds that } \xi_{e_{in},j}^{t+1} < \xi_{e_{out},j}^t. \quad (2)$$

This holds since otherwise, $p_j^{t+1} \cup \{e_{out}\} \setminus \{e_{in}\}$ is a better or not worse deviation.

For every profile (p, ξ), and every sub-optimal player j, let $z_j(p, \xi)$ be the minimal payment of j for a resource that he wishes to exchange in a BR move. Finally, let m_0 be the number of resources with positive load in the initial profile.

Theorem 10. *In uniform matroid games, if ξ_0 is based on fair cost-sharing, and in every BR step a sub-optimal player with minimal $z_j(p, \xi)$ is activated, then a NE is reached after at most $\sum_j \ell_j - m_0$ steps.*

We defer the proof to our full version. We note that for every n, m_0, the above analysis is tight for a symmetric singleton game with n resources. If in the initial profile the players are assigned on $m_0 < n$ resources, then in turn, each activated player will select an empty resource.

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